Analysis 1A — Tutorial 4

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# Introduction

Here is the material to accompany the 4th Analysis Tutorial on the 31st October. Alternative formats can be downloaded by clicking the download icon at the top of the page. As usual, send comments and corrections to [Christian Jones (caj50)](mailto:caj50@bath.ac.uk).

# Lecture Recap

## Sequences and Convergence

Firstly, to discuss anything this week, we need to introduce the idea of a sequence.

Definition 1.1 (Sequence)

A sequence of real numbers is a function $$

Since this notation can get kind of annoying, we instead denote a sequence by . If it’s clear from the context what set we’re indexing over, we can even just simply write a sequence as .

Now, this gives us an infinitely long list of real numbers, and sometimes its interesting to look at the ‘long-term’ behaviour of these lists. This gives rise to the idea of convergence.

Definition 1.2 (Sequence Convergence)

A sequence converges to a real number as , written as either , or if

Loosely speaking, this says that no matter how close you want the sequence to get to , you will always be able to find some point in the sequence after which all points in the sequence will be as close to as you wanted. For an example of this, have a look at this [Desmos link](https://www.desmos.com/calculator/dfkjgg0wzj). For and , you can see that every member of the sequence after the lies within a strip of width around . Have a go at messing with the value of !

Something else we can mention for the definition is its *negation*. Specifically, a sequence does not converge to if

### Useful Sequences

Some (straightforward) results from using the definition include

* As :
* For a real number : as ,
* For with : as

### Two Useful Theorems

Theorem 1.1 (Preservation of Non-Strict Inequalities)

Let and be sequences and let be such that and as . If , then .

There are two good uses for this theorem. The first says that non-negative sequences should have non-negative limits (which is something you might expect). Before we state the second, we mention one more thing, which is **not true**:

Non-theorem:

Let and be sequences and let be such that and as . If , then .

To see why this is false, consider the sequences defined by and . We note that each is strictly less than each corresponding , but we find that

The second reason why Theorem 1.1 is so important, is that it gives us this second theorem[[1]](#footnote-31):

Theorem 1.2 (Uniqueness of Limits)

If is convergent with and as , then .

### Bounded Sequences

Much like we have done with sets, we can formulate a definition which allows us to `trap’ sequences.

Definition 1.3 (Bounded Sequence)

A sequence is bounded if there exists such that .

If you prefer to think diagramatically, this says we can trap the sequence within a strip of width around . More importantly, this leads to the idea that *all convergent sequences are bounded*. Note that this is equivalent to saying that if a sequence is not bounded, then it is not convergent.

### Algebra of Limits

Using the definition to prove all limits would be an incredibly boring way to go through this course. Luckily, there are a few general results we can prove which make our lives so much easier. This is known as the *algebra of limits* (AoL).

Theorem 1.3 (Algebra of Limits)

Let and let and be sequences with and as . Then:

1. ,
2. ,
3. ,
4. If and , .

# Hints

As per usual, here’s where you’ll find the problem sheet hints!

* [H1.] Use the definition! Try and follow a similar format to what we did in tutorials. Make sure to write things logically, and ensure that you’ve satisfied each part of the definition.
* [H2i).] The hint on the sheet will certainly help (can you see the difference of two squares trick here?) The definition is probably the best way to go here. Remember that making a positive denominator smaller will also make the fraction bigger too!
* [H2ii).] Feel free to use AoL here, but make sure to justify why you can use it!
* [H3.] This one is a bit tricky. Firstly, what do you get if you factorise ? Next, you’ll want to use the fact that twice — once to introduce an into the problem, and again to find a point in the sequence after which all of the are positive. Combining all this information should help you prove the required result.

1. Feel free to ignore this footnote, but there are areas of maths where limits are not unique. This is usually in the realm of topology, which you can take in Year 3 [(MA30055)](https://www.bath.ac.uk/catalogues/2022-2023/ma/MA30055.html). Luckily for us, everything behaves nicely, and our limits are unique. [↑](#footnote-ref-31)